Digital Filters

Revision notes by Michael Prior-Jones based on course by Tony Tew

Important things to know:

Digital Filters: benefits & drawbacks

- high stability
- can synthesise very high order filters easily
- extremely versatile: can be used to process any digital data

- complex compared with analogue equivalents
- more expensive than simple analogue filters
- speed of A/D conversion restricts maximum frequency of operation.

Factors determining the performance of a digital filter

- the number of filter taps (determines rate of roll-off)
- the values of the filter coefficients
- whether the filter is recursive (has feedback) or non-recursive (has no feedback).
- the sampling frequency

Nyquist's sampling theorem

In order to avoid aliasing problems, any signal to be sampled should be band-limited to a bandwidth *B*, and the sampling frequency should be set to $f_s > 2B$
The signal may be reconstructed by passing it through a lowpass filter of bandwidth *B*.

Non-recursive filter design

Features of an ideal filter:

- "brick-wall" response (steep roll-off)
- flat passband
- no noise
- zero phase response
- specific break frequency

A zero-phase filter is, sadly, impossible, since that implies that there is no time-delay through the filter. Since all filtering process rely on the storage of energy, this is impossible. Our next-best is a linear-phase filter, which has a constant time-delay through the filter at all frequencies.

Design method for non-recursive filters:

The filter's impulse response is directly specified by its coefficients, so:

- define the desired frequency response
- apply an inverse discrete-time Fourier transform to get the impulse response
- use these values as filter coefficients

The inverse discrete-time Fourier transform is:

$$h[n] = \frac{1}{w_s} \int_{<w_s>} H(w) e^{jwnT_s} dw$$

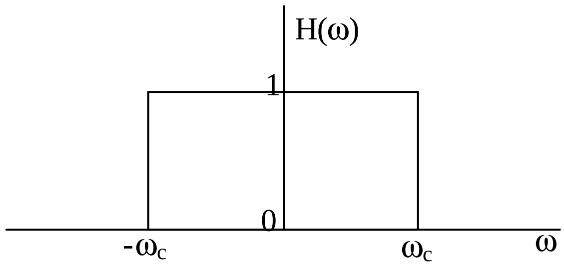
applying the identity $e^x = \cos x + j \sin x$

$$h[n] = \frac{1}{w_s} \int_{<w_s>} H(w)\cos(wnT_s)dw + \frac{j}{w_s} \int_{<w_s>} H(w)\sin(wnT_s)dw$$

h[n] are the filter coefficients, which correspond to the impulse response.
ω_S is the (angular) sampling frequency
H(ω) is the desired filter frequency response.
T_S is the sampling interval.

In order to achieve phase-linearity, we want to ensure that the imaginary part of $h[n]$ is zero. In order to do this we choose H(ω) as a real, even function, because the integral will force sin to zero if this is the case.

H(ω) is chosen as a rectangular function, centred on the origin:



The cut-off frequency is ω_C

When the integral is evaluated, the frequency response of the filter turns out to be rather poor- with low levels of attenuation in the stopband. This is caused by the rectangular window abruptly truncating the sequence of coefficients, and can be cured by using a Hamming window to smooth the transition.
This is done by multiplying each filter coefficient with the appropriate Hamming coefficient, given by:

$$w[n] = 0.54 + 0.46\cos\left(2p\frac{n}{N-1}\right)$$

where

$$\frac{-(N-1)}{2} \le n \le \frac{N-1}{2}$$

The Hamming window provides a flatter passband and greater stopband attenuation, but the roll-off rate is halved.
In fact, the *transition frequency* (the range between the cut-off frequency at the first null) obeys the following equations:

Rectangular window	Hamming window
$f_T = \frac{f_s}{N}$	$f_T = \frac{2 f_s}{N}$

N is the number of filter taps, so these equations can be used to determine the number of taps required to achieve a desired rate of roll-off.

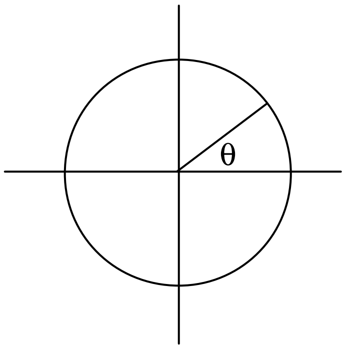
Recursive filter design

This makes use of the z-transform to provide a convenient geometric description of the filter's frequency response.

Simple notch filter design:

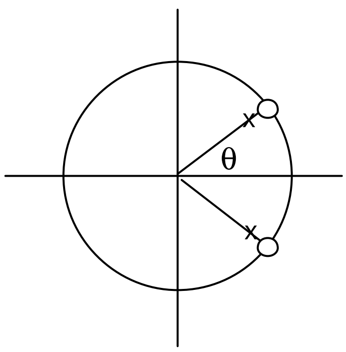
NB- the maths here is the general case, but you can quite easily follow the method yourself with values from an example.

- identify the notch frequency, f_c
- place a zero at this frequency on the unit circle in the z-plane:



$$q = 2p \frac{f_c}{f_s}$$

- add a further zero to give symmetry about the real axis:
- add two poles, just inside the unit circle such that they lie close to the zeroes. The distance of the poles from the centre has the value *r*



- write out the transfer function of these poles and zeroes:

$$H(z) = \frac{(z - e^{jq})(z - e^{-jq})}{(z - re^{jq})(z - re^{-jq})}$$

- rearrange the equation into the quotient of polynomials in z

$$H(z) = \frac{z^2 - ze^{-jq} - ze^{jq} + 1}{z^2 - zre^{-jq} - zre^{jq} + r^2}$$

- apply the identity $\cos x = \dfrac{e^{-jx} - e^{jx}}{2}$

$$H(z) = \frac{z^2 - 2z\cos q + 1}{z^2 - 2rz\cos q + r^2}$$

- multiply through by z^{-2} to produce a polynomial in negative powers of z

$$H(z) = \frac{1 - 2z^{-1}\cos q + z^{-2}}{1 - 2rz^{-1}\cos q + r^2 z^{-2}}$$

- cross-multiply the fraction:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$\therefore Y(z) - 2r\cos(q)z^{-1}Y(z) + r^2 z^{-2}Y(z) = X(z) - 2\cos(q)z^{-1}X(z) + z^{-2}X(z)$$

- perform the inverse z-transform using the timeshift property:

$$A(z)z^{-b} \to a[n-b]$$

$$\therefore y[n] - 2r\cos(q)y[n-1] + r^2 y[n-2] = x[n] - 2\cos(q)x[n-1] + x[n-2]$$

- make $y[n]$ the subject of the equation

$$y[n] = x[n] - 2\cos(q)x[n-1] + x[n-2] + 2r\cos(q)y[n-1] - r^2 y[n-2]$$

- this is the *recurrence expression*- draw the block diagram directly from this.

The Comb filter

This is a special type of filter: it has a frequency response that looks like a row of "teeth". The number of teeth is N, the number of filter taps. In the z-plane it consists of N zeroes spaced equally around the unit circle, starting at $\omega = 0$. There are also N poles placed at the origin.

Pole-zero cancellation method

The comb filter can be used as a basis for designing other filters.

- Draw the unit circle on the z-plane
- Indicate the passband on the unit circle (as for notch filter)
- Choose the value of N such that the comb filters zeroes are spread symmetrically through the passband.
- Identify the zeroes that lie within the passband.
- Cancel these zeroes with conjugate pairs of poles: each zero should have a pole at the same location, and also a conjugate pole to give symmetry in the real axis.
- Revise the number of poles at the origin so that the system still has equal numbers of poles and zeroes.
- Write out the transfer function of the filter and proceed as for the notch filter to get a recurrence expression.
- Draw the block diagram of the filter.